On Functions of Partitioned Matrices with Nondiagonalizable Submatrices

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1. INTRODUCTION

Let M be a square matrix of order q. If the minimum equation of M has only simple roots, then M can be diagonalized and any function $f(z)$ which is defined at the eigenvalues of M may be extended to a function $f(M)$ by means of the Lagrange (Sylvester) formula, which makes use of the idempotent matrix polynomials in M calculated from Lagrange's interpolation formula. If, however, the minimum polynomial of M has repeated roots, then M cannot be diagonalized; nevertheless any function $f(z)$ which has a sufficient number of derivatives may also be extended to a function $f(M)$ by the use of the Lagrange-Hermite formula. This formula is well known (see, for example, [3], p. 83, where it is called the confluent form of Sylvester's Theorem), but it is rarely explained in terms of the Hermite interpolation polynomials. The complete set of Hermite's interpolation polynomials (see, for example, [5], p. 15) was designed to interpolate data when values of a function and some derivatives of it are given at various points. In this paper only a part of the complete set of Hermite's interpolation polynomials will be required. (Cf. [1], p. 443, where the complete set is used to reduce functions of nondiagonalizable matrices to functions of the eigenvalues.) Using these Hermite interpolation polynomials, a reduction formula will be developed for functions of certain partitioned matrices with nondiagonalizable submatrices.

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2. CALCULATION OF FUNCTIONS OF ORDINARY MATRICES BY THE AID OF THE LAGRANGE-HERMITE FORMULA

Let the minimum polynomial of M be $\prod_{i=1}^{s} (x - \lambda_i)^{\mu_i}$, where the λ_i are all distinct and $\sum_{i=1}^{s} \mu_i = r \leqslant q$. Let $H_1(x), H_2(x), \ldots, H_s(x)$ be polynomials of degree $r - 1$ which satisfy the conditions

$$H_i(\lambda_k) = \delta_{ik}, \qquad i, k = 1, 2, 3, \ldots, s;$$

$$H_i^{(\nu)}(\lambda_k) = 0, \qquad \nu = 1, 2, 3, \ldots, \mu_k - 1.$$

If in each of the scalar polynomials $H_1(x), H_2(x), \ldots, H_s(x)$ the scalar variable x is replaced by the matrix M and the scalar unit is replaced by the unit matrix of order q, we obtain s square matrices of order q which will be denoted by $H_1(M), H_2(M), \ldots, H_s(M)$ and will be called "Hermite matrix polynomials." The Hermite matrix polynomials have the fundamental properties

$$\sum_{i=1}^{s} H_i(M) = I, \tag{2.1}$$

$$H_i(M) \cdot H_j(M) = \delta_{ij} H_i(M); \tag{2.2}$$

that is, the matrices $H_1(M), H_2(M), \ldots, H_s(M)$ form a set of orthonormal idempotent projectors. Furthermore

$$(M - \lambda_i I)^{\nu} \cdot H_i(M) = 0 \qquad \text{if} \quad \nu \geqslant \mu_i. \tag{2.3}$$

Having established these properties of the Hermite matrix polynomials, it is a simple matter to develop the Hermite-Lagrange formula:

$$f(M) = \sum_{i=1}^{s} \sum_{\nu=0}^{\mu_i - 1} \frac{f^{(\nu)}(\lambda_i)}{\nu!} (M - \lambda_i I)^{\nu} H_i(A).$$

3. EXTENSION OF THE LAGRANGE-HERMITE FORMULA TO PARTITIONED MATRICES

Let A be a square matrix of order $p \cdot q$ partitioned into p^2 square submatrices of order q. Let each submatrix A_{jk} be a polynomial in a common matrix M:

$$A_{jk} = p_{jk}(M).$$

Let

$$P(x) = [p_{jk}(x)];$$

$P(x)$ is a square matrix of order p with polynomial entries.

If the matrix M has, as above, s distinct eigenvalues of minimal multiplicity μ_i, then each of the submatrices can be expressed by using the Hermite-Lagrange formula

$$A_{jk} = \sum_{i=1}^{s} \sum_{\nu=0}^{\mu_i-1} \frac{1}{\nu!} p_{jk}^{(\nu)}(\lambda_i)(M - \lambda_i I)^\nu H_i(M).$$

The partitioned matrix A may be assembled by using Kronecker (direct) products:

$$A = \sum_{i=1}^{s} \sum_{\nu=0}^{\mu_i-1} \frac{1}{\nu!} P^{(\nu)}(\lambda_i) \times (M - \lambda_i I)^\nu H_i(M).$$

Now A^2 can be calculated as follows:

$$A^2 = \sum_{i=1}^{s} \sum_{j=1}^{s} \sum_{\nu=0}^{\mu_i-1} \sum_{\kappa=0}^{\mu_j-1} \frac{1}{\nu!} \frac{1}{\kappa!} P^{(\nu)}(\lambda_i) P^{(\kappa)}(\lambda_j)$$
$$\times (M - \lambda_i I)^\nu (M - \lambda_j I)^\kappa H_i(M) H_j(M).$$

But $H_i(M)H_j(M) = \delta_{ij}H_i(M)$, so

$$A^2 = \sum_{i=1}^{s} \sum_{\nu=0}^{\mu_i-1} \sum_{\kappa=0}^{\mu_i-1} \frac{1}{\nu!} \frac{1}{\kappa!} P^{(\nu)}(\lambda_i) P^{(\kappa)}(\lambda_i) \times (M - \lambda_i I)^{\nu+\kappa} H_i(M).$$

Taking advantage of property (2.3) and setting $\nu + \kappa = \gamma$,

$$A^2 = \sum_{i=1}^{s} \sum_{\gamma=0}^{\mu_i-1} \left\{ \frac{1}{\gamma!} \sum_{\kappa=0}^{\gamma} \frac{\gamma!}{\kappa!(\gamma - \kappa)!} P^{(\gamma-\kappa)}(\lambda_i) P^{(\kappa)}(\lambda_i) \times (M - \lambda_i I)^\gamma H_i(M) \right\}.$$

Hence

$$A^2 = \sum_{i=1}^{s} \sum_{\gamma=0}^{\mu_i-1} \frac{1}{\gamma!} \{P^2(\lambda)\}_{\lambda=\lambda_i}^{(\gamma)} \times (M - \lambda_i I)^\gamma H_i(M).$$

Extending this result by induction, one can easily show that any analytic function of A may be determined from the same function of $P(x)$. Letting $f\{P(x)\} = \Phi(x)$, we have

$$f(A) = \sum_{i=1}^{s} \sum_{\nu=0}^{\mu_i-1} \frac{1}{\nu!} \Phi^{(\nu)}(\lambda_i) \times (M - \lambda_i I)^\nu H_i(M).$$

This formula is, in fact, valid for any function f which is analytic at the multiple roots of the minimum equation of M and defined at the simple roots. For example, if $P^{-1}(\lambda_i)$ exists for $i = 1, 2, 3, \ldots, s$, then the inverse of A may be calculated from this formula.

4. EXAMPLE

If the system of the second-order differential equations

$$\frac{d^2}{dt^2} y + My = 0$$

is reduced to first order, then the first-order equation can be solved by knowing the function e^A, where

$$A = t \begin{bmatrix} 0 & I \\ -M & 0 \end{bmatrix}.$$

Now the submatrices of A are polynomials in M, and

$$P(x) = t \begin{bmatrix} 0 & 1 \\ -x & 0 \end{bmatrix}.$$

From the power series representation it is found that

$$e^{P(x)} = \begin{bmatrix} \cos \sqrt{x}t & \dfrac{\sin \sqrt{x}t}{\sqrt{x}} \\ -\sqrt{x} \sin \sqrt{x}t & \cos \sqrt{x}t \end{bmatrix}.$$

Let

$$M = \begin{bmatrix} 2 & 2 & 3 \\ 10 & -4 & 5 \\ 5 & -4 & 6 \end{bmatrix}.$$

The eigenvalues of M are $\lambda_1 = 1$ (double), $\lambda_2 = 2$. The corresponding Hermite matrix polynomials are

$$H_1(M) = \begin{bmatrix} 5 & 4 & -8 \\ 15 & 16 & -30 \\ 10 & 10 & -19 \end{bmatrix} \quad \text{and} \quad H_2(M) = \begin{bmatrix} -4 & -4 & 8 \\ -15 & -15 & 30 \\ -10 & -10 & 20 \end{bmatrix}.$$

Thus we have

$$e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \times H_1(M)$$

$$+ \begin{bmatrix} -\dfrac{t \sin t}{2} & \dfrac{t \cos t}{2} - \dfrac{\sin t}{2} \\ -\dfrac{t \cos t}{2} - \dfrac{\sin t}{2} & -\dfrac{t \sin t}{2} \end{bmatrix} \times (M - I)H_1(M)$$

$$+ \begin{bmatrix} \cos \sqrt{2}t & \dfrac{\sin \sqrt{2}t}{\sqrt{2}} \\ -\sqrt{2} \sin \sqrt{2}t & \cos \sqrt{2}t \end{bmatrix} \times H_2(M).$$

5. COMMENTS

In [4], Lynch, Rice, and Thomas invert a certain partitioned matrix whose submatrices commute. No other functions of a partitioned matrix are mentioned, however, and the submatrices are simultaneously diagonalizable.

Williamson [6] gives theorems for the eigenvalues of functions of a partitioned matrix in terms of eigenvalues of the submatrices, the restriction on the submatrices being that they be simultaneously reducible to triangular form.

Egerváry [2] develops methods for the diagonalization of certain partitioned matrices with commutative submatrices. Thus a combination of the results of Williamson and Egerváry permits the computation of functions of a partitioned matrix if the submatrices commute and are diagonalizable.

The results presented here include as special cases the inversion used by Lynch *et al.* and the implied results of Williamson and Egerváry.

6. CONCLUSION

The Lagrange-Hermite formula for functions of a matrix with multiple eigenvalues is extended to partitioned matrices in which all submatrices are polynomials in a common matrix M. The computation of functions of such a matrix is thus reduced to

(i) computation of functions of a smaller matrix which, however, contains a parameter, and

(ii) computation of the Hermite matrix polynomials corresponding to the matrix M.

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